

Two-color QCD in a strong magnetic field: The role of the Polyakov loop

Jens O. Andersen* and Arturo A. Cruz†

Department of Physics, Norwegian University of Science and Technology, Høgskoleringen 5, N-7491 Trondheim, Norway

(Dated: December 6, 2012)

We study two-color QCD in an external magnetic background at finite temperature using the Polyakov-loop extended two-flavor two-color NJL model. At $T = 0$, the chiral condensate is calculated and it is found to increase as a function of the magnetic field B . In the chiral limit the deconfinement transition lies below the chiral transition for nonzero magnetic fields B . At the physical point, the two transitions seem to coincide for field strengths up to $|qB| \approx 5m_\pi^2$ whereafter they split. The splitting between the two increases as a function of B in both the chiral limit and at the physical point. In the range from zero magnetic field and $|qB| = 20m_\pi^2$, the transition temperature for the chiral transition increases by approximately 35 MeV, while the transition temperature for deconfinement is essentially constant.

PACS numbers:

I. INTRODUCTION

The behavior of hadronic matter at finite temperature and density in strong external magnetic fields has received a lot of attention for many years, see for example Ref. [1] for a very recent review. The problem of strongly interacting matter in a strong magnetic background arises in various contexts. For example, magnetars, which are a certain type of neutron stars, have very strong magnetic fields of the order of 10^{10} Tesla (T) [2]. Some of the properties of stars such as the mass-radius relation are determined by the equation of state. The determination of the bulk properties of a Fermi gas in an external magnetic field is therefore important for the understanding of these compact stellar objects. Similarly, large magnetic fields, up to the order of $eB \sim 10^{14-16}$ T, where e is the electric charge of the pion, are being generated in noncentral heavy-collisions at the Relativistic Heavy-Ion Collider (RHIC) and the Large-Hadron Collider (LHC) [3, 4]. The presence of strong magnetic fields may be observed in these experiments via the chiral magnetic effect. This effect is basically the separation of charge in a magnetic background due to the existence of topologically nontrivial configurations in the deconfined phase of QCD [5]. Finally, we mention that strong magnetic fields of the order $10^{14} - 10^{19}$ T may have been present in the early universe during the strong and electroweak phase transitions [6, 7]. The presence of a strong magnetic field at the electroweak phase transition may have implications for baryogenesis, i. e. for the generation of the baryon asymmetry in the universe [8, 9].

Chiral symmetry of the QCD Lagrangian and the

spontaneous breaking of this symmetry in the QCD vacuum is an essential feature of the strong interactions. At $T = 0$, it is expected that a constant magnetic background enhances chiral symmetry breaking if it is present already at $B = 0$ or that it induces chiral symmetry breaking if the symmetry is intact at $B = 0$. This phenomenon is called magnetic catalysis and has been discussed in Refs. [10–17] in the context of the Nambu-Jona-Lasinio (NJL) model, chiral perturbation theory, and QED (note however the recent paper [18] where the authors argue that effects from the neutral mesons might show magnetic inhibition if B is strong enough). The basic mechanism is that neutral quark-antiquarks pair minimize their energy by both aligning their magnetic moments along the direction of the magnetic field [1]. Magnetic catalysis was recently demonstrated on the lattice by Braguta et al [19] in three-color quenched QCD as well as by Bali et al [20] in the context of three-color and 1+1+1 flavor QCD. The results of [20], which are for physical quark masses and extrapolated to the continuum limit, are reproduced very well up to magnetic fields of the order $eB \sim 0.1$ (GeV)² in chiral perturbation theory [15, 16] and up to $eB \sim 0.25$ (GeV)² using the Polyakov-loop extended NJL model (PNJL) [21].

Magnetic catalysis at $T = 0$ gives rise to the expectation that the critical temperature T_c for the chiral transition is an increasing function of the magnetic field B . Indeed, ϕ^4 -theory [22], chiral perturbation theory [23, 24] (However, see also Ref. [25]), the NJL model [26, 27], the PNJL model [21, 28, 29], and the quark-meson (QM) model [27, 30–32] all predict this behavior. Furthermore, the PNJL model also predicts a modest split of approximately 2% between the chiral transition and the deconfinement transition, except for Ref. [33]. In this case the split is of the order 10% and is due to the effects of dimension 8 operators. However, bag-model calculation [34], the Polyakov-loop extended QM calcu-

*Electronic address: andersen@tf.phys.ntnu.no

†Electronic address: arturo.amador@ntnu.no

lation [35], and the large- N_c calculation [36] all predict a decreasing critical temperature as a function of B . In Refs. [34, 35], it is probably related to their treatment of vacuum fluctuations and related renormalization issues.

Turning to lattice simulations, the picture seems to be complicated as well. In Refs. [37, 38], the lattice simulations indicate that the chiral critical temperature is increasing as a function of the magnetic field. In this case, the bare quark masses used correspond to a pion mass in the range $m_\pi = 200 - 480$ MeV, i. e. a very heavy pion. These results have been confirmed by Bali et al [39]. However, for light quark masses that correspond to the physical pion mass of $m_\pi = 140$ MeV, the critical temperature is a decreasing function of the magnetic field B [40]. The basic mechanism seems to be that the magnetic catalysis at $T = 0$ turns into inverse magnetic catalysis [41, 42] for temperatures around the critical temperature T_c . The results suggest that the critical temperature is a nontrivial function of the quark masses.

Two-color QCD is interesting for a number of reasons. For example, in contrast to three-color QCD, one can perform lattice simulations at finite baryon chemical potential μ_B . The reason is that due to the special properties of the gauge group $SU(2)$, the infamous sign problem is absent and thus importance sampling techniques can be used. Moreover, the physics at finite baryon chemical potential is very different from its three-color counterpart: Again due to the properties of the gauge group, two quarks can form a color singlet and so diquarks are found in the spectrum of the chirally broken phase. The diquarks are bosons and finite baryon chemical potential is then the physics of relativistic bosons and their condensation at low temperature. In the chiral limit, the Lagrangian of two-color two-flavor QCD has an $SU(4)$ symmetry. Since this group is isomorphic to $SO(6)$, chiral symmetry breaking can be cast into the form $SO(6) \rightarrow SO(5)$. The Goldstone modes are therefore contained in a single five-plet with the usual three pions, a diquark and an antidiquark as well. Various aspects of the phase diagram of two-color QCD can be found e. g. in Refs. [43–52].

The problem of two-color QCD in a strong magnetic background was first investigated on the lattice by Buidovidovich, Chernodub, Lushevskaya, and Polikarpov [53, 54] in the quenched approximation. Magnetic catalysis at $T = 0$ has been verified and in the chiral limit, the chiral condensate grows linearly for small values of B . This behavior is in qualitative agreement with chiral perturbation theory. Later, lattice simulations have been carried out with dynamical fermions by Ilgenfritz, Kalinowski, Müller-Preussker, Petersson, and Schreiber [55]. Their results seem to indicate that the condensate grows linearly with B in the chiral limit at

$T = 0$. They also found that for all temperatures and fixed bare quark mass, the chiral condensate grows with the magnetic field. This implies that the critical temperature is an increasing function of the magnetic field.

In the present paper, we use the PNJL model to study two-color QCD in a constant magnetic background B at finite temperature and zero baryon chemical potential. The article is organized as follows. In Sec. II, we briefly discuss the PNJL model in a magnetic field and the thermodynamic potential. In Sec. III, we present our numerical results and in Sec. IV, we summarize and conclude.

II. PNJL MODEL AND THERMODYNAMIC POTENTIAL

In this section, we briefly discuss the two-flavor two-color PNJL model. The Euclidean Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2, \quad (1)$$

where the various terms are

$$\mathcal{L}_0 = \bar{\psi}[i\gamma^\mu D_\mu - m_0]\psi, \quad (2)$$

$$\mathcal{L}_1 = G_1 [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 + (\bar{\psi}\boldsymbol{\tau}\psi)^2 + (\bar{\psi}i\gamma_5\boldsymbol{\tau}\psi)^2 + |\bar{\psi}^C\sigma_2\tau_2\psi|^2 + |\bar{\psi}^C\gamma_5\sigma_2\tau_2\psi|^2], \quad (3)$$

$$\mathcal{L}_2 = G_2 [(\bar{\psi}\psi)^2 - (\bar{\psi}i\gamma_5\psi)^2 - (\bar{\psi}\boldsymbol{\tau}\psi)^2 + (\bar{\psi}i\gamma_5\boldsymbol{\tau}\psi)^2 - |\bar{\psi}^C\sigma_2\tau_2\psi|^2 + |\bar{\psi}^C\gamma_5\sigma_2\tau_2\psi|^2], \quad (4)$$

where the quark field ψ is an isospin doublet

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}. \quad (5)$$

The covariant derivative is $D_\mu = \partial_\mu - iqA_\mu - i\sigma_i A_\mu^i$, where A_μ is the gauge field associated with $U(1)$ electromagnetism and A_μ^i is associated with $SU(2)$ color. The covariant derivative is diagonal in flavor space, $q = \text{diag}(2/3, -1/3)|e|$. σ_i ($i = 1, 2, 3$) are the Pauli matrices acting in color space, while τ_i are the Pauli matrices acting in flavor space. m_0 is the mass matrix which is diagonal in flavor space and contains the bare quark masses m_u and m_d . In the following we take $m_u = m_d$. Moreover, ψ^C denotes the charge conjugate of the Dirac spinor, $\psi^C = C\bar{\psi}^T$, where $C = i\gamma^2\gamma^0$. G_1 and G_2 are coupling constants. The interacting part \mathcal{L}_1 is invariant under global $U(4) = SU(4) \times U(1)_A$ transformations while the \mathcal{L}_2 is invariant under global $SU(4)$. One sometimes writes $G_1 = (1 - \alpha)G$ and $G_2 = \alpha G$ and so the parameter α determines the degree of $U(1)_A$ breaking. In the following we choose $\alpha = \frac{1}{2}$.

We next introduce the collective or auxilliary fields

$$\begin{aligned}\sigma &= -2G\bar{\psi}\psi, \pi_i = -2G\bar{\psi}i\gamma_5\tau_i\psi, \rho_i = -2G\bar{\psi}\tau_i\psi, \\ \Delta &= -2G\bar{\psi}i\gamma^5\tau_2\sigma_2\psi^C, \Delta_5 = -2G\bar{\psi}i\tau_2\sigma_2\psi^C, \end{aligned} \quad (6)$$

where σ , π_i , ρ_i , Δ , and Δ_5 have the quantum numbers of a scalar isoscalar, pseudoscalar isovector, scalar isovector, scalar diquark, and pseudoscalar diquarks, respectively. The Lagrangian (1) can then be written compactly as

$$\begin{aligned}\mathcal{L} &= \bar{\psi} [i\gamma^\mu D_\mu - m_0 - \sigma - i\gamma^5\tau_i\pi_i - \tau_i\rho_i] \psi \\ &+ \frac{1}{2} [\Delta^* \bar{\psi}^C i\gamma^5\tau_2\sigma_2\psi + \text{H. c.} + \Delta_5^* \bar{\psi}^C i\tau_2\sigma_2\psi + \text{H. c.}] \\ &- \frac{1}{4G} [\sigma^2 + \pi_i^2 + \rho_i^2 + |\Delta|^2 + |\Delta_5|^2]. \end{aligned} \quad (7)$$

If we use the equation of motion for σ , π_i , τ_i , Δ , and Δ_5 to eliminate the auxilliary fields from the Lagrangian (7), we obtain the original Lagrangian (1).

In pure gauge theory, the Polyakov loop Φ , which is the trace of the Wilson line $l(x) = e^{i\int_0^\beta d\tau \sigma_i A_4^i(x,\tau)}$, i. e. $\Phi = \frac{1}{N_c} \text{Tr} e^{i\int_0^\beta d\tau \sigma_i A_4^i(x,\tau)}$, is an order parameter for deconfinement [56]. Under the center symmetry Z_{N_c} , it transforms as $\Phi \rightarrow e^{2\pi n/N_c} \Phi$, where $n = 0, 1, \dots, N_c - 1$. For $N_c = 2$, this is simply a change of sign and in two-color QCD the Polyakov is purely real. At low temperature, i. e. in the confined phase, we have $\Phi \approx 0$ and in the deconfined phase, we have $\Phi \approx 1$.

In the PNJL model, a constant background temporal gauge field A_4 is introduced via the covariant derivative in Eq. (1) [57, 58]. In Polyakov gauge, the background field is diagonal in color space and, $A_4^i = \theta \delta^{i,3}$, where θ is real. The Wilson line can then be written as $l(x) = \text{diag}(e^{i\beta\theta}, e^{-i\beta\theta})$ and the order parameter Φ reduces to

$$\Phi = \cos(\beta\theta). \quad (8)$$

In order to allow for a chiral condensate, we introduce a nonzero expectation value for the field σ ¹

$$\sigma = -2G\langle\bar{\psi}\psi\rangle + \tilde{\sigma}, \quad (9)$$

where $\tilde{\sigma}$ is a quantum fluctuating fields with vanishing expectation values. To simplify the notation, we introduce the quantity M which is defined by

$$M = m_0 - 2G\langle\bar{\psi}\psi\rangle. \quad (10)$$

Note that the expectation value M is assumed spacetime independent in the remainder of this paper. Eq. (7) is

now bilinear in the quark fields and we integrate them out exactly by performing a Gaussian integral. This gives rise to an effective action for the composite fields. In the mean-field approximation, we neglect the fluctuations of the composite fields and the fermionic functional determinant reduces to

$$\begin{aligned}\Omega_{\text{quark}} &= \frac{(M - m_0)^2}{4G} \\ &- 4N_c \int \frac{d^3p}{(2\pi)^3} \left\{ E_p + T \log [1 + e^{-\beta E_p^\pm}] \right\}, \end{aligned} \quad (11)$$

where $E_p = \sqrt{p^2 + M^2}$ and $E_p^\pm = E_p \pm \theta$. Note that the integral involving E_p is ultraviolet divergent and requires regularization. We will return to this issue below. Summing over \pm , we can write

$$\begin{aligned}\Omega_{\text{quark}} &= \frac{(M - m_0)^2}{4G} - 4N_c \int \frac{d^3p}{(2\pi)^3} \\ &\times \left\{ E_p + T \log [1 + 2\Phi e^{-\beta E_p} + e^{-2\beta E_p}] \right\}. \end{aligned} \quad (12)$$

The interpretation of Eq. (12) is now as follows. For $\Phi \approx 0$, we have confinement and thus a thermal part proportional to $T \log[1 + e^{-2\beta E_p}]$, which corresponds to an excitation of energy $2E_p$, i. e. a bound state. Similarly, for $\Phi \approx 1$, the thermal part is $T \log [1 + 2\Phi e^{-\beta E_p} + e^{-2\beta E_p}] = 2T \log[1 + e^{-\beta E_p}]$ which is the thermal contribution from two degrees of freedom each with energy E_p , i. e. two deconfined quarks.

The complete thermodynamic potential Ω is given by the sum of the contributions from the quarks, Ω_{quark} in Eq. (11) and a contribution from the gluons, Ω_{gauge} , where [48]

$$\Omega_{\text{gauge}} = -bT [24\Phi^2 e^{-\beta a} + \log(1 - \Phi^2)] , \quad (13)$$

where a and b are constants. This form is motivated by the lattice strong-coupling expansion [62]. In the pure gauge theory, we can find an explicit expression for the value of the Polyakov loop, $|\Phi| = \sqrt{1 - \frac{1}{24}e^{\beta a}}$ and so $T_c = a \log 24$. Φ goes to zero in a continuous manner showing that the phase transition is second order in agreement with universality arguments [56].

We next consider this system in a constant magnetic field B along the z -axis. We do this by using the covariant derivative $D_\mu = \partial_\mu - iqa_\mu - i\sigma_i A_\mu^i$, where $a_\mu = \delta_{\mu,2}x_1B$ and $A_\mu^i = \delta^{i,3}\delta_{\mu,4}\theta$. Note that the $SU(4)$ symmetry of the Lagrangian is broken in an external magnetic field due to the different electric charges of the u and d quarks. The remaining symmetry is a $U(1)_A$ symmetry which corresponds to a rotation of the u and d quarks by opposite angles [15]. The chiral condensate breaks this Abelian symmetry and it gives rise to a single Goldstone boson, namely the neutral pion.

¹ Since we consider the case of zero quark chemical potential, the other collective fields have zero expectation value.

The energy eigenvalues of the Dirac equations are in this case given by

$$E_m = \sqrt{p_z^2 + M^2 + (2m + 1 - s)|q_f B|}, \quad (14)$$

where M is the mass of the quark, s is the spin of the quark with electric charge q_f and m denotes the m th Landau level. In Eq. (12), dispersion relation E_p is now changed to E_m and the three-dimensional integral becomes a one-dimensional integral and a sum of Landau levels m . For a quark with charge q_f , we then make the replacements

$$p^2 \rightarrow p_z^2 + (2m - 1 + s)|q_f B|, \quad (15)$$

$$\int \frac{d^3 p}{(2\pi)^3} \rightarrow \frac{|q_f B|}{2\pi} \sum_m \int \frac{dp_z}{2\pi}, \quad (16)$$

where the sum m is over Landau levels and where the prefactor $\frac{|q_f B|}{2\pi}$ takes into account the degeneracy of the Landau levels. The divergent term in Eq. (12) is denoted by $\Omega_{\text{quark}}^{\text{div}}$ and now becomes

$$\Omega_{\text{quark}}^{\text{div}} = -N_c \sum_{f,m,s} \frac{|q_f B|}{\pi} \times \int \frac{dp_z}{2\pi} \sqrt{p_z^2 + M^2 + (2m - 1 + s)|q_f B|}. \quad (17)$$

The integral over p_z as well as the sum over Landau

levels m in Eq. (17) are divergent. We will use zeta-function regularization and dimensional regularization to regulate the divergences. The integral is now generalized to $d = 1 - 2\epsilon$ dimensions using the formula

$$\int \frac{d^d p}{(2\pi)^d} \sqrt{p^2 + M_B^2} = - \left(\frac{e^{\gamma_E} \Lambda^2}{4\pi} \right)^\epsilon \frac{\Gamma(-\frac{d+1}{2})}{(4\pi)^{\frac{d+1}{2}}} M_B^{d+1}, \quad (18)$$

where $M_B^2 = M^2 + (2m + 1 - s)|q_f B|$ and Λ is the renormalization scale in the $\overline{\text{MS}}$ renormalization scheme. This yields

$$\Omega_{\text{quark}}^{\text{div}} = -N_c \left(\frac{e^{\gamma_E} \Lambda^2}{4\pi} \right)^\epsilon \Gamma(-1 + \epsilon) \sum_{f,m,s} \frac{|q_f B|}{4\pi^2} M_B^{2-2\epsilon}, \quad (19)$$

For each flavor f , the sum over m and s can be written as

$$\sum_{m,s} M_B^{2-2\epsilon} = (2q_f B)^{1-2\epsilon} \left[\zeta(-1 + \epsilon, x_f) - \frac{1}{2} x_f^2 \right], \quad (20)$$

where $\zeta(a, x) = \sum_{n=1}^{\infty} \frac{1}{(a+n)^x}$ is the Hurwitz zeta function and $x_f = \frac{M^2}{2|q_f B|}$. Expanding the Hurwitz zeta function in powers of ϵ , we obtain

$$\Omega_{\text{quark}}^{\text{div}} = \frac{N_c}{16\pi^2} \sum_f \left(\frac{\Lambda^2}{2|q_f B|} \right)^\epsilon \left[\left(\frac{2(q_f B)^2}{3} + M^4 \right) \left(\frac{1}{\epsilon} + 1 \right) - 8(q_f B)^2 \zeta^{(1,0)}(-1, x_f) - 2|q_f B| M^2 \log x_f + \mathcal{O}(\epsilon) \right] \quad (21)$$

The first divergence, which is proportional to $(q_f B)^2$ can be removed by wavefunction renormalization of the tree-level term $\frac{1}{2} B^2$ in the free energy. This term is normally omitted since it is independent of the other parameters of the theory. The second divergent term, which is proportional to M^4 is the identical to the divergence that appears for $B = 0$. We can then add and subtract the term

$$\int \frac{d^d p}{(2\pi)^d} \sqrt{p^2 + M^2} = \frac{M^4}{2(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{3}{2} + \mathcal{O}(\epsilon) \right], \quad (22)$$

to Eq. (21) and take the limit $d \rightarrow 3$ in the difference. The divergence is now isolated in the integral on the left-hand-side of Eq. (22). We set $d = 3$ here as well and regulate it by imposing a sharp cutoff Λ in the usual way. The quark thermodynamic potential then becomes

$$\Omega_{\text{quark}} = \frac{(M - m_0)^2}{4G} - \frac{N_c}{4\pi^2} \left[\Lambda \sqrt{\Lambda^2 + M^2} (2\Lambda^2 + M^2) + M^4 \log \frac{\Lambda + \sqrt{\Lambda^2 + M^2}}{M} \right] - \frac{N_c}{16\pi^2} \sum_f (q_f B)^2 [8\zeta'(-1, x_f) - 4(x_f^2 - x_f) \log x_f + 2x_f^2]$$

$$- \sum_{f,m,s} \frac{|q_f B| T}{2\pi} \int_0^\infty \frac{dp_z}{2\pi} \log \left[1 + 2\Phi e^{-\beta E_m} + e^{-2\beta E_m} \right]. \quad (23)$$

The complete thermodynamic potential in a constant magnetic background is the sum of Eqs. (13) and (23) and denoted by Ω . The values of M and the Polyakov loop Φ are found by minimizing Ω with respect to M and Φ , i. e. by solving the gap equations

$$\frac{\partial \Omega}{\partial M} = 0, \quad \frac{\partial \Omega}{\partial \Phi} = 0. \quad (24)$$

Using Eqs. (23), we obtain

$$\begin{aligned} & \frac{(M - m_0)}{2G} - \frac{N_c}{2\pi^2} \left[\Lambda \sqrt{\Lambda^2 + M^2} + \frac{\Lambda^2(2\Lambda^2 + M^2)}{\sqrt{\Lambda^2 + M^2}} + 2M^2 \log \frac{\Lambda + \sqrt{\Lambda^2 + M^2}}{M} \right. \\ & \left. + \frac{M^4}{\Lambda^2 + M^2 + \Lambda \sqrt{\Lambda^2 + M^2}} - M^3 \right] - \frac{N_c}{16\pi^2} \sum_f M |q_f B| \left[8\zeta'(0, x_f) - 2(2x_f - 1) \log x_f + 4x_f \right] \\ & + \frac{N_c}{2\pi^2} \sum_{f,m,s} |q_f B| \int_0^\infty dp_z \frac{M}{E_m} \frac{\Phi e^{-\beta E_m} + e^{-2\beta E_m}}{1 + 2\Phi e^{-\beta E_m} + e^{-2\beta E_m}} = 0, \\ & b\Phi \left[\frac{1}{1 - \Phi^2} - 24e^{-\beta a} \right] - \sum_{f,m,s} \frac{|q_f B|}{2\pi} \int \frac{dp_z}{2\pi} \frac{E^{-\beta E_m}}{1 + 2\Phi e^{-\beta E_m} + e^{-2\beta E_m}} = 0. \end{aligned} \quad (25)$$

$$b\Phi \left[\frac{1}{1 - \Phi^2} - 24e^{-\beta a} \right] - \sum_{f,m,s} \frac{|q_f B|}{2\pi} \int \frac{dp_z}{2\pi} \frac{E^{-\beta E_m}}{1 + 2\Phi e^{-\beta E_m} + e^{-2\beta E_m}} = 0. \quad (26)$$

We notice in particular that $\Phi = 0$ is the only solution to Eq. (26) at $T = 0$ and the PNJL model then reduces to the NJL model.

III. NUMERICAL RESULTS

The PNJL model has five different parameters, namely G , m_0 , and Λ in Ω_{quark} and a and b in Ω_{gauge} . At $T = 0$, $\Omega_{\text{gauge}} = 0$ and so the PNJL model reduces to the NJL model. We can therefore determine the parameters G , m_0 , and Λ separately. For $N_c = 3$, one normally chooses an ultraviolet cutoff Λ and tunes the parameters m_0 and G such that one reproduces the pion mass m_π and the pion decay constant f_π in the vacuum. For $N_c = 2$, we have no experiments to guide us and several different choices have been made [48, 59, 60]. We follow Ref. [48] that uses N_c scaling arguments. Since the pion decay constant f_π is proportional to $\sqrt{N_c}$ and the pion mass is proportional to N_c , we simply rescale the three-color values by $\sqrt{\frac{2}{3}}$ and $\frac{2}{3}$, respectively. This scaling gives $f_\pi = 75.4$ MeV and $m_\pi = 93.3$ MeV. Note that we in the following refer to the case where $m_\pi = 93.3$ MeV as the physical point. With an ultraviolet cutoff of $\Lambda = 657$ MeV, this gives $G = 7.23$ (GeV)⁻² and $m_0 = 5.4$ MeV at the physical point and $G = 7.00$ (GeV)⁻² and $m_0 = 0$ MeV in the chiral limit. The

parameter a in Ω_{gauge} is related to the critical temperature for deconfinement transition in pure-gluon QCD and reads $a = T_c \log 24$. In the pure gauge theory, T_c is independent of the number of colors N_c [61] and so we can use the critical temperature for pure-gluon QCD from lattice calculations with $N_c = 3$, $T_c = 270$ MeV. This yields $a = 858.1$ MeV. The parameter b can be tuned so that the chiral transition takes place at approximately the same temperature as the deconfinement transition. Note, however, that the Polyakov loop is strictly not an order parameter in the presence of dynamical fermions. It is like a crossover and the transition region is defined as a band in which Φ varies. We define the transition region to be the temperatures where $0.4 < \Phi < 0.6$ and the width of the band in the B - T plane tells of fast the crossover is. Nevertheless, we define a deconfinement temperature by the condition $\Phi = \frac{1}{2}$. This gives a curve in the B - T plane and acts a useful guide to the eye. The requirement $\Phi = \frac{1}{2}$ yields $b = (210.5)^3$ (MeV)³. In Fig. 1, we show the chiral condensate and the Polyakov loop in the chiral limit as a function of T/m_π , where $m_\pi = 140$ MeV is the physical pion mass in the vacuum. The chiral condensate (solid) line vanishes at the temperature at which $\Phi = \frac{1}{2}$. Thus for $|qB| = 0$, the two transitions take place at the same temperature as explained above. For comparison, we also plot the chiral condensate in the NJL model as well as the Polyakov

loop in the pure-gluon case, i.e. as derived from the potential Ω_{gauge} in Eq. (13). The chiral transition in the chiral limit is second order for $|qB| = 0$ in the NJL as well as in the PNJL model.

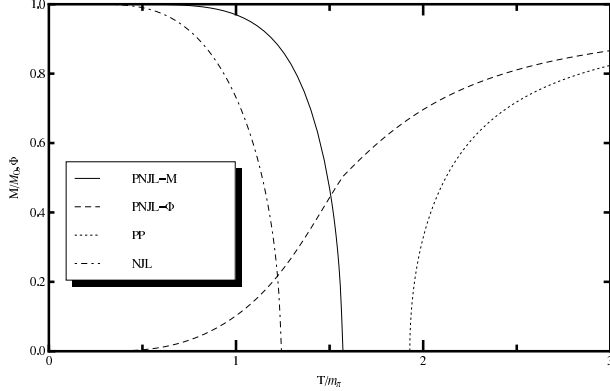


FIG. 1: Normalized chiral condensate and Polyakov loop as a function of T/m_π , where $m_\pi = 140$ MeV.

In Fig. 2, we show the chiral condensate $\langle \bar{\psi}\psi \rangle_B$ normalized to the chiral condensate in the vacuum $\langle \bar{\psi}\psi \rangle_0$ as a function of $|qB|$ in the chiral limit and at the physical point in the vacuum i.e. $T = \mu_B = 0$ in the PNJL model. Note that here and in the following $|qB|$ is measured in units of the pion mass for $N_c = 3$, i. e. $m_\pi = 140$ MeV. Since the effects of the Polyakov vanish in the vacuum this is also the prediction of the NJL model. $\langle \bar{\psi}\psi \rangle_B$ is a monotonically increasing function of $|qB|$ and the system exhibits magnetic catalysis. We notice that magnetic catalysis is stronger in the chiral limit.

We next consider the magnetic moment for a fermion of flavor f . In terms of the spin operator $\Sigma^{\mu\nu} = \frac{1}{2i}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$, it is defined by $\langle \psi_f \Sigma^{\mu\nu} \psi_f \rangle$. In the case of constant magnetic field in the z -direction, only $\Sigma_f \equiv \Sigma^{12}$ is nonzero [63]. Using the properties of the γ -matrices, it can be shown that only the lowest Landau level (LLL) contributes to the expectation value of Σ_f and reads

$$\langle \bar{\psi}_f \Sigma_f \psi_f \rangle_B = \langle \bar{\psi}_f \psi_f \rangle_B^{\text{LLL}}, \quad (27)$$

where the superscript indicates that we include only the lowest Landau level. We then define the polarization by

$$\begin{aligned} \mu_f &= \frac{\langle \bar{\psi}_f \Sigma_f \psi_f \rangle_B}{\langle \bar{\psi}_f \psi_f \rangle_B} \\ &= 1 - \frac{\langle \bar{\psi}_f \psi_f \rangle_B^{\text{HLL}}}{\langle \bar{\psi}_f \psi_f \rangle_B}, \end{aligned} \quad (28)$$

where the superscript HLL indicates that we have included only the higher Landau levels. In Fig. 3, we show

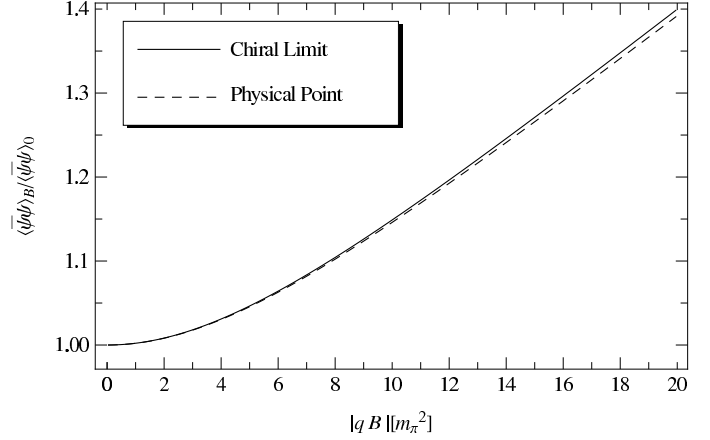


FIG. 2: The chiral condensate $\langle \bar{\psi}\psi \rangle_B$ normalized to the chiral condensate in the vacuum $\langle \bar{\psi}\psi \rangle_0$ as a function of magnetic field $|qB|$ in the chiral limit (solid line) and at the physical point (dashed line) at zero temperature. $m_\pi = 140$ MeV.

the polarization $\mu = \frac{1}{2}(\mu_u + \mu_d)$ at $T = 0$ and in the chiral limit as a function of $|qB|$. As expected, the polarization saturates for large magnetic fields to $\mu^\infty = 1$. In this limit, the higher Landau levels effectively become very heavy (cf. Eq. (14)), they decouple and the LLL dominates the physics. In this limit all the fermions are in the LLL and their spin is pointing in the same direction.

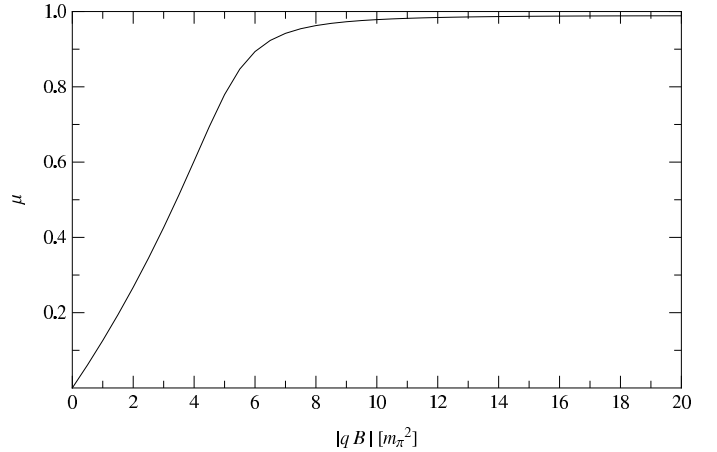


FIG. 3: The polarization μ as a function of $|qB|$ in the chiral limit and at zero temperature. $m_\pi = 140$ MeV.

In Fig. 4, we show the critical temperature for the chiral transition (solid line) and the critical temperature for the deconfinement transition (dashed line) as functions of the magnetic field in the chiral limit for the

PNJL model. The band is defined by $0.4 < \Phi < 0.6$ and shows the transition region for the deconfinement transition. We also show the critical temperature in the NJL model for comparison. The parameter b in Eq. (13) has been tuned such that the two transitions coincide for $B = 0$. We note that there is a splitting between the two transitions and that T_c for the deconfinement is always lower than T_c for the chiral transition and that the gap increases as a function of B . Thus, there should be a phase in which matter is deconfined and chiral symmetry is broken. The splitting was also observed in Ref. [29] where the authors coupled the Polyakov loop to linear sigma model with quarks with $N_c = 3$ colors. Moreover, while T_c for the chiral transition increases by more than 20% from $B = 0$ to $|qB| = 20m_\pi^2$, T_d for the deconfinement transition is hardly affected. The width of the band is approximately 30 MeV. In contrast, the lattice simulations for $N_c = 2$ reported in Ref. [55] indicate that the critical temperature for deconfinement coincide with that of the chiral transition. Finally, we note that the determination of a and b has changes the critical temperature for the chiral transition dramatically. The increase of T_c at $B = 0$ is approximately 35 MeV and is fairly constant up to $|qB| = 20$ MeV. In order to compare the chiral transition at finite magnetic field in the NJL and PNJL model, i. e. the effects of the Polyakov loop, there might be other ways of determining a and b . For example, one could force the deconfinement transition and the chiral transition in the PNJL model to take place at the same temperature for $B = 0$ and force it to coincide with the chiral transition in the NJL model as well. This way of determining the parameters in Ω_{gauge} would not require the input from lattice simulations.

In Fig. 5, we show the thermodynamic potential $\Omega - \Omega_0$ divided by f_π^4 in the chiral limit for four different temperatures and $|qB| = 20m_\pi^2$. For each temperature, we also give the value of the Polyakov loop. The critical temperature for the chiral transition is $T_c = 220$ MeV and from the long-dashed line we see that transition is second order. Since the value of the Polyakov loop for $T = 220$ MeV is $\Phi = 0.68$, we conclude that the deconfinement transition has already taken place.

In Fig. 6, we show the thermodynamic potential $\Omega - \Omega_0$ divided by f_π^4 as a function of Φ in the chiral limit for four different temperatures and $|qB| = 20m_\pi^2$. For each temperature, we also give the value of the chiral condensate M . At $T = 197$ MeV, we find that the minimum of the effective potential occurs for $\Phi = \frac{1}{2}$ which defines the critical temperature for the deconfinement transition in the chiral limit. At this temperature, $M = 286.2$ MeV and so we are still in the chirally broken phase. For $T = 250$ MeV, the chiral condensate is vanishing and so we are in the chirally symmetric deconfined phase. In the chiral limit, the chiral transition is always second

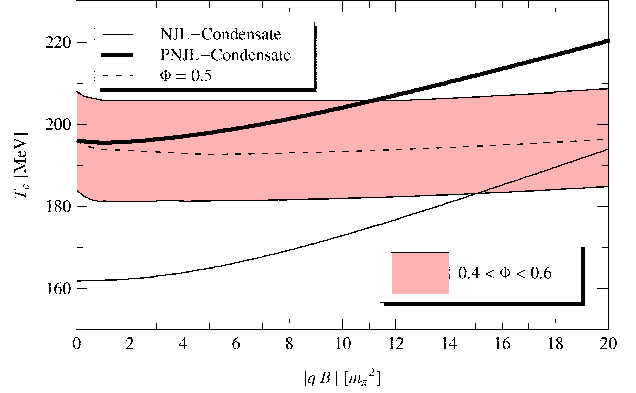


FIG. 4: Critical temperature for the chiral transition (solid line) and critical temperature for the deconfinement transition (dashed line) as functions of magnetic field $|qB|$ in the chiral limit for the PNJL model. The band is defined by $0.4 < \Phi < 0.6$. We also show the critical temperature for the chiral transition (thin line) in the NJL model. $m_\pi = 140$ MeV.

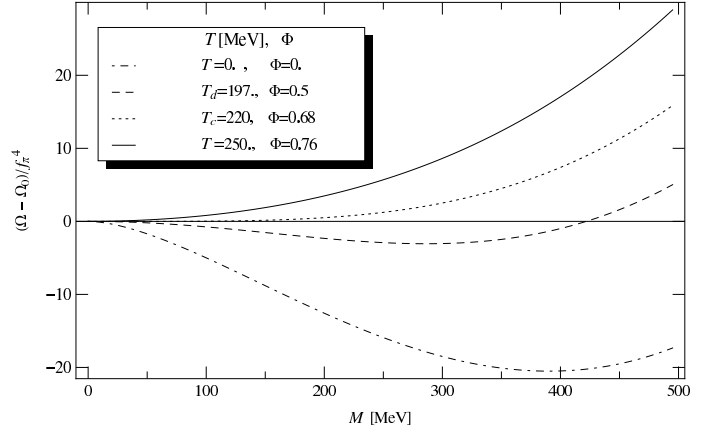


FIG. 5: Thermodynamic potential $\Omega - \Omega_0$ divided by f_π^4 as a function of M in the chiral limit for four different temperatures and $|qB| = 20m_\pi^2$, where $m_\pi = 140$ MeV. See main text for details.

order.

At the physical point, the chiral transition is a crossover and there is no well-defined critical temperature T_c at which the chiral condensate vanishes. However, one can define a pseudo-critical temperature by the inflection point of the chiral condensate as a function of temperature. One can also define the transition region by the temperature range where M varies between 0.4 and 0.6. This range depends on the magnetic field and gives rise to a band in the B - T plane. This is shown as the dark band in Fig. 7. In the same manner, we define

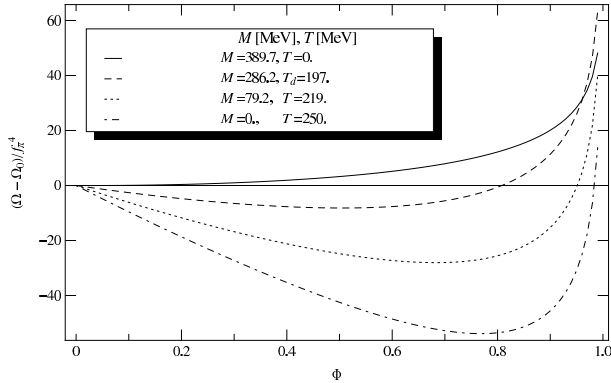


FIG. 6: Thermodynamic potential $\Omega - \Omega_0$ divided by f_π^4 as a function of Φ in the chiral limit for four different temperatures and $|qB| = 20m_\pi^2$, where $m_\pi = 140$ MeV. See main text for details.

the crossover transition for the deconfinement transition by the temperature range where $0.4 < \Phi < 0.6$. This is shown as the light band in Fig. 7. Moreover, as a guide to the eye, we also show the lines where $M = \frac{1}{2}$ and $\Phi = \frac{1}{2}$, respectively. The curves indicate that the two transitions coincide for magnetic fields up to $|qB| \approx 5m_\pi^2$ and after that they split. The deconfinement transition is always taking place first. Finally, we note that the band of the chiral transition is much narrower than that of the deconfinement transition (approximately $T = 10$ MeV versus approximately $T = 30$ MeV) and is so significantly faster. In Fig. 8, we show the normalized ther-

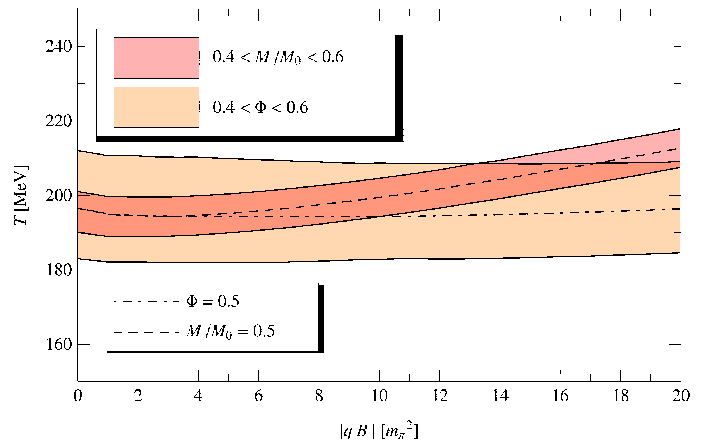


FIG. 7: Transition region at the physical point as a function of $|qB|$. $m_\pi = 140$ MeV. See main text for details.

modynamic potential Ω/f_π^4 as a function of M and Φ in the chiral limit with $|qB| = 20m_\pi^2$ for different temperatures. From left to right $T = 100$ MeV, $T = 200$ MeV, $T = 219$ MeV, and $T = 250$ MeV. For $T = 100$ MeV, we are in the confined and the chirally broken phase, while for $T = 200$ MeV, the value of Φ is slightly above $\frac{1}{2}$ ($\Phi = \frac{1}{2}$ for $T = 197$ MeV). The minimum is still for nonzero M and so chiral symmetry is broken. For $T = 219$ MeV, M/M_0 is still nonzero ($T_c = 220$ MeV) and so we are just below the chiral transition.

IV. CONCLUDING REMARKS

In the present paper, we have considered two-color two-flavor QCD in a constant magnetic background using the PNJL model. In the chiral limit, the chiral transition is always second order with mean-field critical exponents. This is in agreement with universality arguments. Our result for T_c for the chiral transition as a function of B for $N_c = 2$ is qualitatively the same as that for $N_c = 3$, namely that it increases. For the PNJL model with lack of confinement and no dynamical gauge fields, we do not expect a qualitative different behavior. Whether the different results for T_c for $N_c = 2$ and $N_c = 3$ obtained on the lattice is due to the different gauge groups and the very different physics in the vacuum (diquarks rather baryons as bound states) or taking the continuum limit is an open question. For $N_c = 3$, it seems that taking the continuum limit is essential to

obtain the decrease of T_c as function of B and so the only way to settle the issue for $N_c = 2$ is to perform a careful study of the continuum limit in this case as well. Either way, given the lattice results for $N_c = 3$ it is clear that all the model calculations to date fail at temperatures T around T_c and in particular does not incorporate that magnetic catalysis is turned into inverse magnetic catalysis around the transition. This is independent of whether it is a mean-field calculation or goes beyond using e.g. functional renormalization group techniques [30, 31]. However, in the recent paper [18], the authors argue that fluctuations from neutral meson can exhibit magnetic inhibition, i. e. inverse magnetic catalysis if the magnetic field B is sufficiently strong. This may provide a mechanism that explains the recent lattice results.

It is also interesting to note that the magnetic field hardly affects the critical temperature for deconfinement.

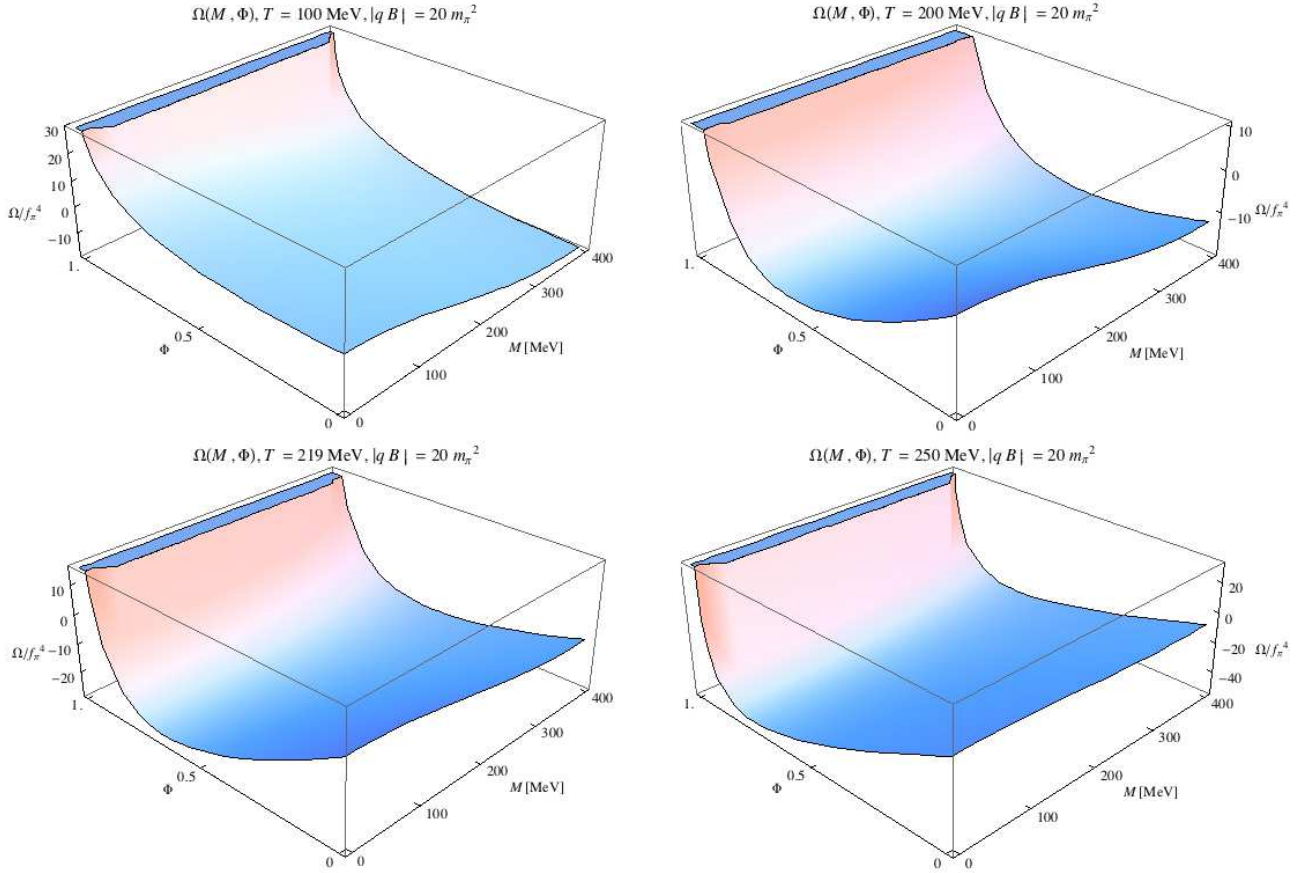


FIG. 8: The normalized thermodynamic potential Ω/f_π^4 as a function of M and Φ for four different temperatures and $|qB| = 20m_\pi$. From left to right $T = 100$ MeV, $T = 200$ MeV, $T = 219$ MeV, and $T = 250$ MeV. See main text for details.

ment. This is in line with the observation of [29]. Moreover, our results seem to indicate that the two transitions coincide at the physical point up to fairly large values of the magnetic field. $|qB| \approx 4m_\pi^2$.

Finally, we would like to comment on the role of quantum fluctuations and related renormalization issues. In a one-loop calculation of the effective potential it is possible to separate the vacuum contributions from the thermal contributions. In some case, it therefore makes sense to investigate the role of the vacuum fluctuations. For example, in the QM, it is customary to treat the bosons at tree level and the fermions at the one-loop level. In this case it was shown in Ref. [64] for $B = 0$ that the order of the phase transition depends whether the zero-temperature fluctuations are included or not; if they are,

the chiral transition is second order and if they are not, it is first order. The same effect of the vacuum fluctuations were found in the entire μ_B - T plane in strong magnetic fields in [32]. In contrast to the QM model with quarks, this question does not make sense in the (P)NJL model. The reason is that chiral symmetry breaking in the (P)NJL is always a loop effect in contrast to the QM model where it is built into the tree-level potential. In a similar manner, Ref. [65] finds that a crossover transition (for $N_c = 3$) at the physical point remains a crossover at finite magnetic field B in an NJL model calculation. This is in contrast to Ref. [29] where it is found that strong magnetic fields turn the crossover into a first-order transition. In their work, the authors use the QM model and renormalize by subtracting the

fermionic vacuum fluctuations at $B = 0$.

Acknowledgments

J. O. A. would like to thank Tomas Brauner for useful discussions.

-
- [1] Lect. Notes Phys. "Strongly interacting matter in magnetic fields" (Springer), edited by D. Kharzeev, K. Landsteiner, A. Schmitt, and H.-U. Yee.
 - [2] R. C. Duncan and C. Thompson, *Astrophys. J.* **392** L9 (1992).
 - [3] V. Skokov, A. Y. Illarionov, and V. Toneev, *Int. J. Mod.Phys. A* **24** 5025, (2009).
 - [4] A. Bzdak and V. Skokov, *Phys.Lett. B* **710**, 171 (2012).
 - [5] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, *Nucl. Phys. A* **803**, 227 (2008).
 - [6] T. Vachaspati, *Phys. Lett. B* **265**, 258 (1991).
 - [7] K. Enqvist and P. Olesen, *Phys. Lett. B* **319**, 178 (1993).
 - [8] M. Laine K. Kajantie, M. Laine, J. Peisa, K. Rummukainen, M. E. Shaposhnikov, *Nucl. Phys. B* **544**, 357 (1999).
 - [9] A. De Simone, G. Nardini, M. Quiros, and A. Riotto, *JCAP* 1110, 030 (2011).
 - [10] S. P. Klevansky and R. H. Lemmer, *Phys. Rev. D* **39**, 3478, (1989).
 - [11] K.G. Klimenko, *Z. Phys. C* **54**, 323 (1992).
 - [12] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, *Phys. Rev. Lett.* **73**, 3499 (1994).
 - [13] V. P. Gusynin, V.A. Miransky, and I. A. Shovkovy, *Nucl. Phys. B* **462**, 249 (1996).
 - [14] D. Ebert, K. G. Klimenko, M. A. Vdovichenko, and A. S. Vshivtsev, *Phys. Rev. D* **61** 025005 (1999).
 - [15] I. Shushpanov and A. V. Smilga, *Phys. Lett. B* **402** 351, (1997).
 - [16] T. D. Cohen, D. A. McGady, and E. S. Werbos, *Phys. Rev. C* **76**, 055201 (2007).
 - [17] K. Fukushima and J. M. Pawłowski, *Phys. Rev. D* **86**, 076013 (2012).
 - [18] K. Fukushima and Y. Hidaka, arXiv: 1209.1319V1 [hep-ph].
 - [19] V. V. Braguta, P. V. Buividovich, T. Kalaydzhyan, S. V. Kuznetsov, and M. I. Polikarpov, PoS LATTICE2010, 190 (2010), *Phys. Atom. Nucl.* **75**, 488 (2012).
 - [20] G. S. Bali, F. Bruckmann, G. Endrődi, Z. Fodor, S.D. Katz, and A. Schafer, *Phys. Rev. D* **86**, 071502 (2012).
 - [21] R. Gatto and M. Ruggieri, *Phys. Rev. D* **82**, 054027 (2010).
 - [22] D. C. Duarte, R. L. S. Farias, and R. O. Ramos, *Phys. Rev. D* **84**, 083525 (2011).
 - [23] N. O. Agasian, *Phys. Lett. B* **488**, 39 (2000).
 - [24] J. O. Andersen, *Phys. Rev. D* **86**, 025020 (2012); *JHEP* **1210**, 005 (2012).
 - [25] N. O. Agasian and S. M. Fedorov, *Phys. Lett. B* **663**, 445 (2008).
 - [26] S. S. Avancini, D. P. Menezes, M. B. Pinto, and C. Providencia, *Phys.Rev. D* **85** 091901 (2012).
 - [27] G. N. Ferrari, A. F. Garcia, and M. B. Pinto, *Phys. Rev. D* **86**, 096005 (2012).
 - [28] K. Fukushima, M. Ruggieri, and R. Gatto, *Phys. Rev. D* **81**, 114031 (2010).
 - [29] E. S. Fraga and A. J. Mizher, *Phys. Rev. D.* **78**, 025016 (2008).
 - [30] V. Skokov, *Phys. Rev. D* **85**, 03426 (2012).
 - [31] J. O. Andersen and A. Tranberg, *JHEP* **1210**, 002 (2012).
 - [32] J. O. Andersen and R. Khan, *Phys. Rev. D* **85**, 065026 (2012).
 - [33] R. Gatto and M. Ruggieri, *Phys. Rev. D* **83**, 034016 (2011).
 - [34] E. S. Fraga and L. F. Palhares, *Phys. Rev. D* **86**, 016008 (2012).
 - [35] A. J. Mizher, M. N. Chernodub and E. S. Fraga, *Phys. Rev. D* **82**, 105016 (2010).
 - [36] E. S. Fraga, J. Noronha, and L. F. Palhares, e-Print: arXiv:1207.7094.
 - [37] M. D'Elia, S. Mukherjee, and F. Sanfilippo, *Phys. Rev. D* **82**, 051501 R (2010).
 - [38] M. D'Elia and F. Negro, *Phys. Rev. D* **83**, 114028 (2011).
 - [39] G. Bali, G. Endrődi, and F. Bruckmann, Private communications.
 - [40] G. S. Bali, F. Bruckmann, G. Endrődi, Z. Fodor, S. D. Katz, S. Krieg, A. Schafer, and K. K. Szabo, *JHEP* **1202**, 044 (2012).
 - [41] T. Inagaki, D. Kimura, and T. Murata, *Prog. Theor. Phys.* **111**, 371 (2004).
 - [42] F. Preis, Anton Rebhan, and Andreas Schmitt. *JHEP* **1103**, 033 (2011).
 - [43] J. B. Kogut, M. A. Stephanov, and D. Toublan, *Phys.Lett. B* **464**, 183 (1999).
 - [44] K. Splittorff, D.T. Son, and M. A. Stephanov, *Phys. Rev. D* **64**, 016003 (2001).
 - [45] P. Cea, L. Cosmai, M. D'Elia, and A. Papa *JHEP* **0702**, 066 (2007.)
 - [46] S. Hands, S. Kim, J-I. Skullerud, *Phys. Rev. D* **81** 091502 (2010).
 - [47] T. Kanazawa, Tilo Wettig, N. Yamamoto, *JHEP* **0908**, 003 (2009).
 - [48] T. Brauner, K. Fukushima, and Y. Hidaka *Phys. Rev. D* **80**, 074035 (2009); Erratum-ibid. *D* **81**, 119904 (2010).

- [49] J. O. Andersen and T. Brauner, Phys. Rev. D **81**, 096004 (2010).
- [50] T. Zhang, T. Brauner, A. Kurkela, A. Vuorinen, JHEP **1202**, 139 (2012).
- [51] N. Strodthoff, B.-J. Schaefer, L. von Smekal, Phys. Rev. D **85**, 074007 (2012).
- [52] S. Imai, H. Toki, and W. Weise e-Print: arXiv:1210.1307 [nucl-th].
- [53] P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya, and M. I. Polikarpov, Phys. Lett. B **682**, 484 (2010).
- [54] P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya, and M. I. Polikarpov, Nucl. Phys. B **826**, 313 (2010).
- [55] E.-M. Ilgenfritz, M. Kalinowski, M. Müller-Preussker, B. Petersson, and A. Schreiber, Phys. Rev. D **85**, 114504 (2012).
- [56] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B **210**, 423 (1982).
- [57] K. Fukushima, Phys. Lett. B **591**, 277 (2004).
- [58] E. Megias, E. Ruiz Arriola, and L.L. Salcedo, Phys. Rev. D **74**, (2006) 065005; *ibid* **74**, 114014 (2006).
- [59] C. Ratti and W. Weise, Phys. Rev. D **70**, 054013 (2004).
- [60] G.-F. Sun, L. He, P. Zhuang, Phys. Rev. D **75**, 096004 (2007).
- [61] C. Sasaki, B. Friman, and K. Redlich, Phys. Rev. D **75**, 074013 (2007).
- [62] K. Fukushima, Phys. Rev. D **77**, 114028 (2008).
- [63] M. Frasca and M. Ruggieri, Phys. Rev. D **83**, 094024 (2011).
- [64] V. Skokov, B. Friman, E. Nakano, K. Redlich, and B.-J. Schaefer, Phys. Rev. D **82**, 034029 (2010).
- [65] J. K. Boomsma and D. Boer, Phys. Rev. D **81**, 074005 (2010).